

BETTI SPLITTING VIA COMPONENTWISE LINEAR IDEALS

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ABSTRACT. A monomial ideal I admits a Betti splitting $I = J + K$ if the Betti numbers of I can be determined in terms of the Betti numbers of the ideals J, K and $J \cap K$. Given a monomial ideal I , we prove that $I = J + K$ is a Betti splitting of I , provided J and K are componentwise linear, generalizing a result of Francisco, Hà and Van Tuyl. If I has a linear resolution, the converse also holds. We apply this result recursively to the Alexander dual of vertex-decomposable, shellable and constructible simplicial complexes and to determine the graded Betti numbers of the defining ideal of three general fat points in the projective space.

1. INTRODUCTION

Our aim is to pursue the spirit of Eliahou and Kervaire [3], Francisco, Hà and Van Tuyl [7] in order to find suitable decomposition of the Betti table of a monomial ideal, possibly available for recursive procedures.

Let \mathbb{k} be a field and let $I \subseteq R = \mathbb{k}[x_1, \dots, x_n]$ be a monomial ideal. Consider $J, K \subseteq I$ monomial ideals such that the set of minimal monomial generators of I is the disjoint union of the sets of minimal monomial generators of J and K . We say that $I = J + K$ is a *Betti splitting* of I if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K), \text{ for all } i, j \in \mathbb{N},$$

where $\beta_{i,j}(-)$ denotes the graded Betti numbers of a minimal R -free graded resolution.

This approach was used for the first time by Eliahou and Kervaire [3], giving an explicit formula for the total Betti numbers of stable ideals. Fatabbi [5] developed the theory for graded Betti numbers. Many authors wrote papers applying the Eliahou-Kervaire technique to the resolution of special classes of monomial ideals (see e.g. [4], [5], [6], [8], [17], [19], [23], [24], [25]). Francisco, Hà and Van Tuyl proved in [7, Corollary 2.4] that if J and K have a linear resolution, then $I = J + K$ is a Betti splitting of I . In Section 3 we generalize this result, proving that $I = J + K$ is a Betti splitting of I , provided J and K are componentwise linear (Theorem 3.3). If I has a linear resolution, the converse also holds (Proposition 3.1).

Componentwise linear ideals have been extensively studied (see e.g. [6], [8], [10], [11]). Stable ideals, ideals with linear quotients and ideals with linear resolution are examples of componentwise linear ideals.

In Section 4 we apply the theory of Betti splittings to the Alexander dual ideal I_{Δ}^* of a simplicial complex Δ . By the Stanley-Reisner correspondence, squarefree monomial ideals correspond to simplicial complexes (see e.g. [10]). This is an important bridge between Commutative Algebra and Combinatorics. In particular, by Hochster's formula [15], the graded Betti numbers of I_{Δ}^* reflect geometric and topological information on Δ .

As a consequence of Theorem 3.3, we recover a result due to Moradi and Kosh-Ahang [17], proving that the Alexander dual of a vertex-decomposable simplicial complex admits a particular kind of splitting, the so-called x_i -splitting (Corollary 4.3). In Corollary 4.5 we prove a Betti splitting result for shellable and constructible simplicial complexes, showing that in general they do not admit x_i -splitting (Example 4.4).

A further application is an extension of a result proved by Valla in [25]. By using a recursive approach we can compute explicitly the graded Betti numbers of the defining ideal of three general fat points in the projective space (Corollary 5.2 and Corollary 5.3).

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2. PRELIMINARIES

Let \mathbb{k} be a field, $R = \mathbb{k}[x_1, \dots, x_n]$, \mathfrak{m} the maximal homogeneous ideal of R , $I \subseteq R$ an homogeneous ideal. Denote by $\beta_{ij}(I) = \dim_{\mathbb{k}} \operatorname{Tor}_i^R(I, \mathbb{k})_j$ the *graded Betti numbers* of I . In the following we omit the superscript R . Given a monomial ideal I , let $G(I)$ be the minimal system of monomial generators of I and $\operatorname{indeg}(I)$ be the lowest degree of a generator in $G(I)$.

Definition 2.1. (Francisco, Hà, Van Tuyl, [7]) Let I, J and K be monomial ideals such that $I = J + K$ and $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Then $J + K$ is a *Betti splitting* of I if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K), \text{ for all } i, j \in \mathbb{N}.$$

The previous definition can be given in terms of the vanishing of some Tor modules maps, as stated in the following result.

Proposition 2.2. (Francisco, Hà and Van Tuyl, [7, Proposition 2.1]) *Let I, J and K be monomial ideals such that $I = J + K$ and $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Consider the short exact sequence*

$$0 \rightarrow J \cap K \rightarrow J \oplus K \rightarrow I \rightarrow 0$$

and the corresponding long exact sequence of Tor modules:

$$\cdots \rightarrow \operatorname{Tor}_{i+1}(I, \mathbb{k})_j \rightarrow \operatorname{Tor}_i(J \cap K, \mathbb{k})_j \xrightarrow{\phi_{i,j}} \operatorname{Tor}_i(J, \mathbb{k})_j \oplus \operatorname{Tor}_i(K, \mathbb{k})_j \rightarrow \operatorname{Tor}_i(I, \mathbb{k})_j \rightarrow \cdots$$

Then the following are equivalent:

- (i) $I = J + K$ is a Betti splitting;
- (ii) $\phi_{i,j} = 0$, for all $i, j \in \mathbb{N}$.

If I is generated in degree d we say that I has a d -linear resolution if $\beta_{i,i+j}(I) = 0$ for every $i, j \in \mathbb{N}$ and $j \neq d$. When the context is clear, we simply write that I has a linear resolution. The *Castelnuovo-Mumford regularity* of I is defined by $\operatorname{reg}(I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\}$. An ideal I generated in degree d has a d -linear resolution if and only if $\operatorname{reg}(I) = d$.

Let I, J and K be monomial ideals, with $I = J + K$ and $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Francisco, Hà and Van Tuyl proved in [7, Corollary 2.4] that if J and K have a linear resolution, then $I = J + K$ is a Betti splitting. In Section 3 we generalize this result assuming J and K componentwise linear.

We recall some definitions and results that will be useful later. These hold also in a more general setting, but from now on we assume R to be the standard graded polynomial ring with coefficients in a field \mathbb{k} (see [12],[18], [21] and [22] for more details).

Componentwise linear ideals have been introduced by Herzog and Hibi in [11]. Denote by $I_{<j>}$ the ideal generated by all the homogeneous polynomials of degree j belonging to I . In the monomial case, $I_{<j>}$ is simply the ideal generated by all monomials of degree j belonging to I .

Definition 2.3. A homogeneous ideal $I \subseteq R$ is called *componentwise linear* if $I_{<j>}$ has a linear resolution, for every $j \in \mathbb{N}$.

Notice that we cannot detect if an ideal is componentwise linear from its graded Betti numbers. In the following example we show an ideal I that is not componentwise linear but with the same graded Betti numbers of a componentwise linear ideal.

Example 2.4. ([14, Example 5.5]) Let $I, J \subseteq \mathbb{k}[x, y, z]$ be the ideals $I = (x^4, x^3y, x^2y^2, x^3z, xy^2z, xyz^2, xy^4, x^2z^3, y^4z)$ and $J = (x^4, x^3y, x^2y^2, xy^3, y^4, x^3z, x^2yz^2, x^2z^3, xy^2z^2)$. Using CoCoA [2], one can see that I and J have the same graded Betti numbers. The ideal J is stable hence componentwise linear, while $I_{<4>}$ has not a linear resolution, thus I is not componentwise linear.

Linearity defect was introduced by Herzog and Iyengar in [12] and measures how far a resolution is from being linear.

Let M be a finitely generated graded R -module and \mathbb{F} its minimal graded free resolution over R . Let $\text{lin}^R(\mathbb{F})$ be the chain complex obtained by \mathbb{F} replacing by zero each entry of degree greater than one in the matrices of the differentials of \mathbb{F} .

Definition 2.5. Let M be a finitely generated graded R -module M . The *linearity defect* of M is defined by

$$\text{ld}_R(M) := \sup\{i : H_i(\text{lin}^R(\mathbb{F})) \neq 0\},$$

where $H_i(\text{lin}^R(\mathbb{F}))$ denotes the i -th homology of the chain complex $\text{lin}^R(\mathbb{F})$.

We close this section by stating a special case of a useful characterization of componentwise linear modules due to Römer (see also [26, Proposition 4.9]).

Theorem 2.6. (Römer,[21, Theorem 3.2.8]) *For any homogeneous ideal $I \subseteq R$, I is componentwise linear if and only if $\text{ld}_R(I) = 0$.*

3. RESULTS

Let I, J and K be monomial ideals, with $I = J + K$ and $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Francisco, Hà and Van Tuyl proved in [7, Corollary 2.4] that if J and K have a linear resolution, then $I = J + K$ is a Betti splitting. Provided I with a linear resolution, the converse also holds, as stated in the following result.

Proposition 3.1. *Let d be a positive integer, I be a monomial ideal with a d -linear resolution, $J, K \neq 0$ monomial ideals such that $I = J + K$, $G(I) = G(J) \cup G(K)$ and $G(J) \cap G(K) = \emptyset$. Then the following facts are equivalent:*

- (i) $I = J + K$ is a Betti splitting of I ;
- (ii) J and K have d -linear resolutions.

If this is the case, then $J \cap K$ has a $(d + 1)$ -linear resolution.

Proof. Assume (i) holds. Then $\beta_{i,i+j}(I) = \beta_{i,i+j}(J) + \beta_{i,i+j}(K) + \beta_{i-1,i+j}(J \cap K)$ for all $i, j \geq 0$. Let $i \geq 0$. For $j \neq d$ we have $\beta_{i,i+j}(I) = \beta_{i,i+j}(J) = \beta_{i,i+j}(K) = 0$, thus J and K have a d -linear resolution.

Conversely one has that (ii) implies (i), by [7, Corollary 2.4].

By [7, Corollary 2.2] we have $\text{reg}(J \cap K) \leq \text{reg}(I) + 1 = d + 1$. Since $\text{indeg}(J \cap K) \geq d + 1$, then $\text{reg}(J \cap K) \geq \text{indeg}(J \cap K) \geq d + 1$, thus $J \cap K$ has a $(d + 1)$ -linear resolution. \square

Notice that there exist ideals with a linear resolution that do not admit any Betti splitting (see Example 5.4).

Now we extend [7, Corollary 2.4], assuming J and K componentwise linear ideals. In the proof we use the following key-lemma.

Lemma 3.2. (Nguyen, [18, Lemma 2.8(ii)]) *Let $M \rightarrow P$ be an R -linear map between finitely generated R -modules. If for some $k \geq \text{ld}_R(P) + 1$, the map $\text{Tor}_{k-1}(M, \mathbb{k}) \rightarrow \text{Tor}_{k-1}(P, \mathbb{k})$ is zero, then the map*

$$\text{Tor}_i(M, R/\mathfrak{m}^s) \rightarrow \text{Tor}_i(P, R/\mathfrak{m}^s)$$

is zero for all $i \geq k$ and all $s \geq 0$.

Theorem 3.3. *Let I, J and K be monomial ideals such that $I = J + K$ and $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. If J and K are componentwise linear, then $I = J + K$ is a Betti splitting of I .*

Proof. By Proposition 2.2, it suffices to prove that all the maps

$$\text{Tor}_i(J \cap K, \mathbb{k}) \xrightarrow{\phi_i} \text{Tor}_i(J \oplus K, \mathbb{k}) = \text{Tor}_i(J, \mathbb{k}) \oplus \text{Tor}_i(K, \mathbb{k})$$

are zero for all $i \geq 0$. Since J and K are componentwise linear, by Theorem 2.6 we have $\text{ld}_R(J) = \text{ld}_R(K) = 0$. Then $\text{ld}_R(J \oplus K) = 0$. Since $G(I)$ is the disjoint union of $G(J)$ and $G(K)$, hence $J \cap K \subseteq \mathfrak{m}J$ and $J \cap K \subseteq \mathfrak{m}K$. Then the map

$$\text{Tor}_0(J \cap K, \mathbb{k}) \xrightarrow{\phi_0} \text{Tor}_0(J \oplus K, \mathbb{k})$$

is zero. By Lemma 3.2 with $k = s = 1$ the result follows. \square

In view of Proposition 3.1, the assumptions on J and K in Theorem 3.3 cannot be weakened in general, without further assumptions.

Clearly the converse of Theorem 3.3 does not hold in general, as shown in the following example.

Example 3.4. Let $I \subseteq \mathbb{k}[x_1, x_2, x_3, x_4, x_5, x_6]$ be the monomial ideal defined by

$$I = (x_4x_5x_6, x_1x_2x_6, x_1x_3x_4).$$

Define $J = (x_4x_5x_6)$ and $K = (x_1x_2x_6, x_1x_3x_4)$. It is easy to check that $I = J + K$ is a Betti splitting of I . In fact the graded Betti numbers of I, J, K and $J \cap K$ are given by

$$0 \rightarrow R(-6) \rightarrow R(-5)^3 \rightarrow R(-3)^3 \rightarrow I.$$

$$0 \rightarrow R(-3) \rightarrow J.$$

$$0 \rightarrow R(-5) \rightarrow R(-3)^2 \rightarrow K.$$

$$0 \rightarrow R(-6) \rightarrow R(-5)^2 \rightarrow J \cap K.$$

Nevertheless K is not componentwise linear, since K is generated in one degree and has no linear resolution.

Several examples and Proposition 3.1 suggest the following question.

Question: Assume I componentwise linear. Does the converse of Theorem 3.3 hold?

4. BETTI SPLITTING FOR SIMPLICIAL COMPLEXES

In this section we present some applications of Theorem 3.3 to simplicial complexes. For more definitions about simplicial complexes, their properties and the Stanley-Reisner correspondence we refer to [10, Chapter 1] and [16, Chapter 3] and references over there.

Definition 4.1. An *abstract simplicial complex* Δ on n vertices is a collection of subsets of $\{1, \dots, n\}$, called *faces*, such that if $F \in \Delta$, $G \subseteq F$, then $G \in \Delta$.

Denote a face $A = \{i_1, \dots, i_q\}$ by $A = [i_1, \dots, i_q]$, with $i_1 < \dots < i_q$. A *facet* is a maximal face of Δ with respect to the inclusion of sets. Denote by $\mathcal{F}(\Delta)$ the collection of facets of Δ . A simplicial complex is called *pure* if all the facets of Δ have the same cardinality.

The *Alexander dual ideal* I_Δ^* of Δ is defined by

$$I_\Delta^* = (x_{\overline{F}} \mid F \in \mathcal{F}(\Delta)), \text{ where } \overline{F} := \{1, \dots, n\} \setminus F \text{ and } x_{\overline{F}} = \prod_{i \in \overline{F}} x_i.$$

A decomposition $\Delta = \Delta_1 \cup \Delta_2$, such that $\mathcal{F}(\Delta)$ is the disjoint union of $\mathcal{F}(\Delta_1)$ and $\mathcal{F}(\Delta_2)$ induces a decomposition $I_\Delta^* = I_{\Delta_1}^* + I_{\Delta_2}^*$. We call $\Delta = \Delta_1 \cup \Delta_2$ a *Betti splitting* of Δ if $I_\Delta^* = I_{\Delta_1}^* + I_{\Delta_2}^*$ is a Betti splitting of I_Δ^* . Note that all the Alexander dual ideals involved are computed with respect to the vertices of Δ .

In the following diagram (see e.g. [16]) we recall the hierarchy of some properties of (possibly non-pure) simplicial complexes.

$$(4.1) \quad \text{vertex decomposable} \rightarrow \text{shellable} \rightarrow \text{constructible} \rightarrow \text{sequentially Cohen-Macaulay}.$$

It can be proved ([13], see also [10, Theorem 8.2.20]) that a simplicial complex Δ is sequentially Cohen-Macaulay if and only if I_Δ^* is componentwise linear.

All these three properties are defined recursively and for this reason we can apply Theorem 3.3 to I_Δ^* .

In [7, Theorem 2.3] Francisco, Hà and Van Tuyl give conditions on J, K and $J \cap K$ forcing $I = J + K$ to be a Betti splitting of I . The splitting given in Theorem 3.3 is not a consequence of [7, Theorem 2.3], as shown in the following example.

Example 4.2. Let \mathbb{k} be a field of characteristic zero and let $I_\Delta^* \subseteq \mathbb{k}[x_1, \dots, x_{12}]$ be the Alexander dual ideal of the simplicial complex Δ whose set of facets:

$$\mathcal{F}(\Delta) = \{[1, 2, 3, 4], [1, 2, 3, 12], [3, 4, 6], [3, 4, 5], [4, 5, 6], [3, 5, 6], [5, 6, 7], [5, 7, 8], [4, 9], [9, 10], [10, 11], [6, 9], [8, 12]\}.$$

Let $\Delta = \Delta_1 \cup \Delta_2$, with $\mathcal{F}(\Delta_1) = \{[1, 2, 3, 4], [1, 2, 3, 12], [3, 4, 5], [3, 4, 6], [4, 9]\}$ and $\mathcal{F}(\Delta_2) = \{[3, 5, 6], [4, 5, 6], [5, 6, 7], [5, 7, 8], [6, 9], [8, 12], [10, 11], [9, 10]\}$.

Let J_i be the Alexander dual ideal of Δ_i , for $i = 1, 2$. Both Δ_1 and Δ_2 are shellable (use *Macaulay* [9]). Then J_1 and J_2 are componentwise linear. By Theorem 3.3, $I_\Delta^* = J_1 + J_2$ is a Betti splitting of I . Nevertheless the assumptions of [7, Theorem 2.3] are not satisfied since $\beta_{1,11}(J_1 \cap J_2) > 0$ and both $\beta_{1,11}(J_1)$ and $\beta_{1,11}(J_2)$ are not zero.

In the next result we focus on a special splitting of a monomial ideal I . Let x_i be a variable of R . Let J be the ideal generated by all monomials of $G(I)$ divided by a x_i and let K be the ideal generated by the remaining monomials of $G(I)$. If $I = J + K$ is a Betti splitting, we call $I = J + K$ a x_i -splitting of I .

We recover the following known result.

Corollary 4.3. ([17, Theorem 2.8, Corollary 2.11]) *If Δ is a vertex decomposable simplicial complex then there exists $i \in V(\Delta)$ such that I_Δ^* admits x_i -splitting.*

Proof. Since Δ is vertex decomposable, there exists a vertex $i \in V(\Delta)$ such that $\text{del}_\Delta(i)$ and $\text{link}_\Delta(i)$ are both vertex decomposable. By [17, Lemma 2.2] we have $I_\Delta^* = x_i I_{\text{del}_\Delta(i)}^* + I_{\text{link}_\Delta(i)}^*$. A vertex decomposable simplicial complex is sequentially Cohen-Macaulay by Diagram 4.1. Then $x_i I_{\text{del}_\Delta(i)}^*$ and $I_{\text{link}_\Delta(i)}^*$ are componentwise linear. The statement follows from Theorem 3.3. \square

In Corollary 4.3, vertex decomposability cannot be replaced by shellability or constructibility, as it is shown in the next example.

Example 4.4. Consider the simplicial complex Δ whose set of facets is

$$\mathcal{F}(\Delta) = \{[2, 3, 4], [2, 4, 7], [1, 2, 7], [1, 6, 7], [2, 3, 5], [1, 2, 5], [1, 2, 6], [2, 3, 6], [3, 5, 6], [5, 6, 7], [4, 5, 7], [1, 4, 5], [1, 3, 4], [1, 3, 7], [3, 5, 7], [4, 6, 7], [4, 6, 12], [6, 11, 12], [6, 8, 12], [8, 9, 12], [6, 8, 9], [6, 9, 10], [6, 10, 11], [9, 10, 11], [8, 9, 11], [4, 11, 12], [4, 8, 11], [4, 9, 12], [4, 9, 10], [4, 8, 10], [8, 10, 12], [10, 11, 12]\}.$$

The given order of the facets of Δ is indeed a shelling, thus Δ is shellable (use *Macaulay* [9]). By Diagram 4.1, I_Δ^* is componentwise linear. Since I_Δ^* is generated in degree 9, I has actually a 9-linear resolution. Consider the splitting $I_\Delta^* = x_i J + K$, for every $1 \leq i \leq 12$. The resolution of $x_i J$ is *not* linear, for every $1 \leq i \leq 12$. By Proposition 3.1, I does not admit x_i -splitting.

Let Δ be a constructible simplicial complex. Although in general I_Δ^* does not admit any x_i -splitting, it admits a Betti splitting, as a consequence of Theorem 3.3.

Corollary 4.5. *Let Δ be a constructible simplicial complex. Then I_Δ^* admits Betti splitting.*

Proof. By definition of constructible complexes, there exist two constructible simplicial complexes Δ_1 and Δ_2 such that $\Delta = \Delta_1 \cup \Delta_2$ and $\mathcal{F}(\Delta)$ is the disjoint union of $\mathcal{F}(\Delta_1)$ and

$\mathcal{F}(\Delta_2)$. Then $I_{\Delta_1}^*$ and $I_{\Delta_2}^*$ are componentwise linear ideals, by Diagram 4.1. By Theorem 3.3, $I_{\Delta}^* = I_{\Delta_1}^* + I_{\Delta_2}^*$ is a Betti splitting of I_{Δ}^* . \square

In the pure case the previous result is [19, Corollary 3.4]. With the same notations in the proof of Corollary 4.5, for shellable complexes a more precise result holds: Δ_2 consists of a single facet, because in this case I_{Δ}^* has linear quotients (see [1]).

Notice that Corollary 4.5 does not hold for (sequentially) Cohen-Macaulay simplicial complexes, as shown in the following example. We present an ideal I that *does not admit Betti splitting at all*, in characteristic different from two. To our knowledge this is the first example in literature of an ideal that does not admit any Betti splitting (see [1, Example 5.31] for a characteristic-free example).

Example 4.6. Let \mathbb{k} be a field of $\text{char}(\mathbb{k}) \neq 2$. Let Δ be the following 6-vertex triangulation with 10 facets of the real projective plane, due to Reisner [20].

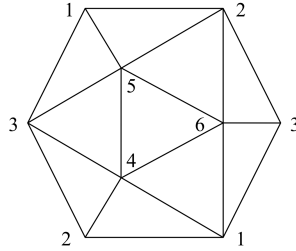


FIGURE 1. A triangulation of real projective plane.

The graded Betti numbers of I_{Δ}^* are given by

$$0 \rightarrow R(-5)^6 \rightarrow R(-4)^{15} \rightarrow R(-3)^{10} \rightarrow I_{\Delta}^*.$$

Francisco, Hà and Van Tuyl in [7] pointed out that I_{Δ}^* does not admit x_i -splitting (in this case the Stanley-Reisner ideal I_{Δ} and I_{Δ}^* coincide). Actually I_{Δ}^* does not admit *any* Betti splitting. Using CoCoA [2] and Macaulay [9], we checked all the possible $\sum_{i=1}^5 \binom{10}{i} = 637$ pairs of ideals J and K such that $G(I_{\Delta}^*)$ is the disjoint union of $G(J)$ and $G(K)$. In each case, at least one between J and K has no linear resolution. Then, by Proposition 3.1, $I_{\Delta}^* = J + K$ is not a Betti splitting.

5. THE RESOLUTION OF THE IDEAL OF THREE GENERAL FAT POINTS IN \mathbb{P}^{n-1}

Let $X = \{(P_1, a), (P_2, b), (P_3, c)\}$ be the 0-dimensional scheme consisting of three general fat points in \mathbb{P}^{n-1} , with $n \geq 4$ and $1 \leq a \leq b \leq c$. After a change of coordinates, we may assume that $P_1 = [1 : 0 : \dots : 0]$, $P_2 = [0 : 1 : 0 : \dots : 0]$ and $P_3 = [0 : 0 : 1 : 0 : \dots : 0]$. Then the defining ideal of X is

$$I_{n,a,b,c} = (x_2, \dots, x_n)^a \cap (x_1, x_3, \dots, x_n)^b \cap (x_1, x_2, x_4, \dots, x_n)^c \subseteq R = \mathbb{k}[x_1, \dots, x_n].$$

In the case of two fat points we denote $I_{n,0,b,c}$ by $I_{n,b,c}$, with $b \leq c$. If $a = b = c$ we denote $I_{n,a,a,a}$ by $I_{n,a}$.

Francisco proved in [6] that the defining ideal of the zero-dimensional schemes of $r \leq n+1$ general fat points in \mathbb{P}^n is componentwise linear. In general the ideals $I_{n,a,b,c}$ are not stable (even if $a = b = c = 1$). Valla [25, Corollary 3.5] computed the graded Betti numbers of the defining ideal of two general fat points in \mathbb{P}^n , $n \geq 2$, by using a Betti splitting argument. We prove a splitting result for $I_{n,a,b,c}$ and, as a consequence, we give a recursive procedure to compute the graded Betti numbers of $I_{n,a,b,c}$ in the case $a \neq c$.

Theorem 5.1. *Let $n \in \mathbb{N}$, $n \geq 4$. Let $X = \{(P_1, a), (P_2, b), (P_3, c)\}$ be the 0-dimensional scheme defined by three general fat points in \mathbb{P}^{n-1} , with $1 \leq a \leq b \leq c$. Assume $c \neq a$. Then $I = I_{n,a,b,c}$ admits x_1 -splitting.*

Proof. Let $J = x_1 I_{n,a,b-1,c-1}$ and $K = (x_3, \dots, x_n)^b \cap (x_2, x_4, \dots, x_n)^c$. We show that $I = J + K$.

Let $f \in G(I)$. Assume $f \in (x_1)$. Then $f \in (x_1) \cap I$. Since $(x_1) \cap (x_1, x_3, \dots, x_n)^b = x_1(x_1, x_3, \dots, x_n)^{b-1}$ and $(x_1) \cap (x_1, x_2, x_4, \dots, x_n)^c = x_1(x_1, x_2, x_4, \dots, x_n)^{c-1}$, it follows

$$(x_1) \cap I = x_1(x_2, \dots, x_n)^a \cap x_1(x_1, x_3, \dots, x_n)^{b-1} \cap x_1(x_1, x_2, x_4, \dots, x_n)^{c-1} = x_1 I_{n,a,b-1,c-1} = J.$$

Let $f \notin (x_1)$, then $f \in (x_2, \dots, x_n)^a \cap (x_3, \dots, x_n)^b \cap (x_2, x_4, \dots, x_n)^c$. Since $(x_3, \dots, x_n) \subseteq (x_2, x_3, \dots, x_n)$ and $a \leq b$, then $f \in (x_3, \dots, x_n)^b \cap (x_2, x_4, \dots, x_n)^c = K$. For the other inclusion, note that $J = (x_1) \cap I \subseteq I$ and $K = (x_3, \dots, x_n)^b \cap (x_2, x_4, \dots, x_n)^c = (x_2, \dots, x_n)^a \cap (x_3, \dots, x_n)^b \cap (x_2, x_4, \dots, x_n)^c \subseteq I$.

We prove now that $G(I) = G(J) \cup G(K)$. Let $g \in G(I)$. Then $g \in J$ or $g \in K$. Assume $g \in J$. If $g \notin G(J)$, there would be $m \in G(J)$ such that $g \in (m)$. Since $J \subseteq I$ and $g \in G(I)$, this is a contradiction. The proof for K is the same, then the first inclusion is clear.

For the reverse inclusion, we first prove that $K \subseteq \mathfrak{m} I_{n,a,b-1,c-1}$. Let $h \in G(K)$ be a monomial. Assume first that there is a variable x_i , with $x_i | h$ and $4 \leq i \leq n$. Then

$$h \in x_i[(x_3, \dots, x_n)^{b-1} \cap (x_2, x_4, \dots, x_n)^{c-1}] \subseteq \mathfrak{m}[(x_3, \dots, x_n)^{b-1} \cap (x_2, x_4, \dots, x_n)^{c-1}].$$

Since $a \leq c-1$ and $(x_2, x_4, \dots, x_n) \subseteq (x_2, \dots, x_n)$ hence

$$h \in \mathfrak{m}[(x_2, \dots, x_n)^a \cap (x_3, \dots, x_n)^{b-1} \cap (x_2, x_4, \dots, x_n)^{c-1}] \subseteq \mathfrak{m} I_{n,a,b-1,c-1}.$$

Otherwise $h = x_2^c x_3^b = x_2 x_3 (x_2^{b-1} x_3^{c-1}) \in \mathfrak{m} I_{n,a,b-1,c-1}$.

Let $g \in G(J)$. By definition of J , there is $s \in G(I_{n,a,b-1,c-1})$ such that $g = x_1 s$. If $g \notin G(I)$, then there are $h \in G(I)$ and a monomial $r \in R$ such that $g = x_1 s = hr$. Since $G(I) \subseteq G(J) \cup G(K)$ and $g \in G(J)$, then $h \in G(K)$. Hence $x_1 | r$ and $s = r_1 h$, where $r_1 = \frac{r}{x_1}$. This is a contradiction, because $h \in \mathfrak{m} I_{n,a,b-1,c-1}$.

Let $g \in G(K)$. If $g \notin G(I)$, then there is $m \in G(I)$ such that $g \in (m)$. Since $g \in G(K)$, then $m \in J = (x_1) \cap I$. Hence $x_1 | g$, a contradiction.

Clearly one has $G(J) \cap G(K) = \emptyset$. By [6, Theorem 4.6], J and K are componentwise linear. In view of Theorem 3.3, we conclude that $I = J + K$ is a Betti splitting of I . \square

Notice that, in general, the splitting of Theorem 5.1 is not a particular case of [7, Theorem 2.3] (see for instance the case $n = 4$, $a = b = 1$, $c = 2$).

In the next corollary we compute explicitly the graded Betti numbers of $I_{n,a,b,c}$ in the case $a + b \leq c$ by a recursive procedure.

Corollary 5.2. *Let $n \in \mathbb{N}$, $n \geq 4$. Let $X = \{(P_1, a), (P_2, b), (P_3, c)\}$ be the 0-dimensional scheme consisting of three general fat points in \mathbb{P}^{n-1} , with $1 \leq a \leq b \leq c$ and $I = I_{n,a,b,c}$. Assume $a + b \leq c$. Then*

$$\beta_{i,i+c}(I) = \beta_{i,i+c-b}(I_{n,a,c-b}) + \sum_{r=0}^{b-1} [\beta_{i,i+c-r}(I_{n-1,b-r,c-r}) + \beta_{i-1,i+c-r-1}(I_{n-1,b-r,c-r})].$$

$$\beta_{i,j}(I) = \begin{cases} \binom{n-2}{i} \left[\binom{n-3+c+a-j+i}{n-3} + \binom{n-3+c+b-j+i}{n-3} \right] & \text{if } c+1+i \leq j \leq a+c+i. \\ \binom{n-2}{i} \binom{n-3+c+b-j+i}{n-3} & \text{if } a+c+1+i \leq j \leq b+c+i. \\ 0 & \text{if } j \geq b+c+1+i. \end{cases}$$

Proof. Since $a + b \leq c$, the assumptions of Theorem 5.1 are fulfilled, then $I_{n,a,b,c}$ admits x_1 -splitting. Let J and K be as in the proof of Theorem 5.1. Clearly $J \cap K = x_1 K$. Note that $\beta_{i,j}(J) = \beta_{i,j-1}(I_{n,a,b-1,c-1})$ and $\beta_{i-1,j}(J \cap K) = \beta_{i-1,j-1}(K)$, for each $i, j \geq 0$. We remark that, after a relabeling of the variables, K is the ideal of two general fat points in \mathbb{P}^{n-2} , i.e. $K = I_{n-1,b,c}$. Then

$$\beta_{i,j}(I) = \beta_{i,j-1}(I_{n,a,b-1,c-1}) + \beta_{i,j}(I_{n-1,b,c}) + \beta_{i-1,j-1}(I_{n-1,b,c}).$$

Since $c - r \neq a$, for $0 \leq r \leq c - a - 1$ and $b - 1 \leq c - a - 1$, we can apply the same argument recursively to $I_{n,a,b-r,c-r}$, for $0 \leq r \leq b - 1$, to get

$$(1) \quad \beta_{i,j}(I) = \beta_{i,j-b}(I_{n,a,c-b}) + \sum_{r=0}^{b-1} [\beta_{i,j-r}(I_{n-1,b-r,c-r}) + \beta_{i-1,j-1-r}(I_{n-1,b-r,c-r})].$$

The result follows by [25, Corollary 3.5] and by the relation $\sum_{r=h}^s \binom{r}{c} = \binom{s+1}{c+1} - \binom{h}{c+1}$. \square

Theorem 5.1 allows us to apply a recursive procedure for computing the Betti numbers of $I = I_{n,a,b,c}$ also in the case $a + b > c$ and $c \neq a$. This formula has the limit that the Betti numbers of $I_{n,k}$, $k \in \mathbb{N}$, could be involved. These ideals are studied in [5]. An explicit formula for the graded Betti numbers of $I_{n,k}$ is given only for $k = 2$ [6, Proposition 3.2] and $k = 3$ [6, Proposition 3.3].

Corollary 5.3. *Let $n \in \mathbb{N}$, $n \geq 4$. Let $X = \{(P_1, a), (P_2, b), (P_3, c)\}$ be the 0-dimensional scheme consisting of three general fat points in \mathbb{P}^{n-1} , with $1 \leq a \leq b \leq c$, $a + b > c$ and $c \neq a$. Let $I = I_{n,a,b,c}$ be the defining ideal of X . Set $k = a + b - c$,*

$$B_{n,a,b,c}^{i,j} := \binom{n-3+c+a-j+i}{n-3} + \binom{n-3+c+b-j+i}{n-3} - 2 \binom{n-3+a+b-j+i}{n-3}, \text{ for } i, j \in \mathbb{N},$$

and

$$\gamma_{s,t}^{n,i} := \beta_{i,i+t}(I_{n-1,s,t}) + \beta_{i-1,i+t-1}(I_{n-1,s,t}), \text{ for } s, t \in \mathbb{N}.$$

Then

$$\beta_{i,i+c}(I) = \beta_{i,i+k}(I_{n,k}) + \sum_{r=0}^{c-a-1} \gamma_{b-r,c-r}^{n,i} + \sum_{r=0}^{c-b-1} \gamma_{a-r,a-r}^{n,i}.$$

$$\beta_{i,j}(I) = \begin{cases} \beta_{i,j+k-c}(I_{n,k}) + \binom{n-2}{i} B_{n,a,b,c}^{i,j} & \text{if } c+1+i \leq j \leq a+b+i. \\ \binom{n-2}{i} \left[\binom{n-3+c+a-j+i}{n-3} + \binom{n-3+c+b-j+i}{n-3} \right] & \text{if } a+b+1+i \leq j \leq a+c+i. \\ \binom{n-2}{i} \binom{n-3+c+b-j+i}{n-3} & \text{if } a+c+1+i \leq j \leq b+c+i. \\ 0 & \text{if } j \geq c+b+1+i. \end{cases}$$

Proof. Note that $c < a + b \leq 2c$ and $c \geq 2$. By Theorem 5.1, $I_{n,a,b,c}$ admits x_1 -splitting. The main difference with Corollary 5.2 is that $c - r \neq a$ for $0 \leq r \leq c - a$ but $c - a < b$. Then, following the proof of Corollary 5.2 and using Equation (1) we get

$$\beta_{i,j}(I) = \beta_{i,j+a-c}(I_{n,a,a+b-c,a}) + \sum_{r=0}^{c-a-1} [\beta_{i,j-r}(I_{n-1,b-r,c-r}) + \beta_{i-1,j-1-r}(I_{n-1,b-r,c-r})].$$

We focus our attention only on the first term $\beta_{i,j+a-c}(I_{n,a,a+b-c,a})$ of the equation. Now one has $a + b - c \leq a$. If $b = c$ the claim follows. If $b < c$, then $a + b - c \neq a$ and the assumptions of Theorem 5.1 are satisfied. Since we consider the new order of the ideals of the intersection given by multiplicities, then $I_{n,a,a+b-c,a}$ admits x_2 -splitting. One has $a - r \neq a + b - c$ for $0 \leq r \leq c - b - 1$. Using Equation (1) and $k = a + b - c$, we get

$$\beta_{i,j+a-c}(I_{n,k,a,a}) = \beta_{i,j+k-c}(I_{n,k}) + \sum_{r=0}^{c-b-1} [\beta_{i,j+a-c-r}(I_{n-1,a-r,a-r}) + \beta_{i-1,j+a-c-r-1}(I_{n-1,a-r,a-r})].$$

By [6, Proposition 3.1], the ideal $I_{n,k}$ is generated in degree at most $2k$. The statement follows [25, Corollary 3.5]. \square

We prove that in Theorem 5.1 the assumption $c \neq a$ is essential.

Example 5.4. Consider three double points in \mathbb{P}^3 . The defining ideal $I = I_{4,2}$ admits the decomposition $I = J + K$ of Theorem 5.1, where $J = x_1 I_{4,2,1,1}$ and $K = (x_3, x_4)^2 \cap (x_2, x_4)^2$. Unfortunately $G(I) \neq G(J) \cup G(K)$, since we have $x_1 x_4^2 \in G(J)$ that is *not* a minimal generator of I . The same problem arise if we choose x_2 or x_3 .

It can be proved [1] that, in general, $I_{n,a}$ admits x_n -splitting. More precisely, we have $I_{n,a} = x_n I_{n,a-1} + I_{n-1,a}$, with $G(I_{n,a}) = G(x_n I_{n,a-1}) \cup G(I_{n-1,a})$ and $G(x_n I_{n,a-1}) \cap G(I_{n-1,a}) = \emptyset$, but we are not able to take advantage from this decomposition, since the resolutions of both $I_{n,a-1}$ and $I_{n-1,a}$ are in general unknown.

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